Remarks on the History and Philosophy of Mathematics

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REMARKS ON THE HISTORY AND PHILOSOPHY OF MATHEMATICS

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The history and philosophy of mathematics bear only on the periphery of the work of most mathematicians whether they are teachers or researchers. As a consequence there builds up a 'received view' of these subjects, a body of dogma which remains for the most part unarticulated. This dogma influences our work even if the influence seems small and comes from afar. The primary purpose of this essay is to bring the received dogma into view and examine it. In the process we will see that it is more influential than we supposed. Implicit in the entire undertaking is my conviction that an understanding of the history and philosophy of mathematics is interesting in itself and can contribute to our effectiveness as teachers.

Relevance. Before beginning the remarks that constitute the main body of this essay I should sketch an argument to make plausible my assertion that they can be relevant to us as teachers. My aim here is not so much to build a tight case as to suggest a possibility. I believe that the remarks themselves will strengthen the case.

George Pólya argues persuasively (in *Mathematical Discovery*, [**31**], see especially section 5 of the preface, section 14.2, and comment 14.5) that the primary aim of mathematical instruction is to teach problem solving. Obviously there is room for discussion about what constitutes a problem and the word 'problem' can take on a meaning so wide as to render it useless for thinking. But let us accept Pólya's assertion as a provisional guideline.

A problem is a question posed for consideration or solution. I usually approach a problem guided by the analogy with a journey. One must know where one is, where one wants to go, and then try to find a route. In some situations the first two elements are clear and we may even develop an algorithmic procedure for solving problems of a particular type without having to give them further thought. Solving linear equations in one unknown is an example of this sort. In other cases it requires a lot of thought about the destination before there is any hope of trying to find a route. Many proofs are of this sort; it is difficult to get a clear conception of what is to be proved. The problem of disposing of nuclear wastes is also of this sort. It appears that elementary instruction (including the college level 'service course') often assumes an overly narrow conception of a problem (first type only) and then makes things worse by going directly to the algorithm instead of allowing it to reveal itself as the labor saving device it is.

To return to the instructional problem, we believe that we understand the destination. The point of departure is not as clear. In each class we must make an effort to find out *where the students are* and then to chart a course. For the most part this means: choose what we will use to try to motivate them. A great variety of analogies are available for motivation. Their usefulness is related to some philosophical issues which I will discuss later. For now it will suffice to glance at what all too often passes for motivation.

Of course the purpose of motivation is to help propel the student along the path from where he is now to his destination. But what is his destination? Are we to ask him? We can hardly hope to tell him; after all, how does one motivate a student to achieve an end he hasn't accepted as his own? A goal and the process of working toward it are a dialectical pair. The vaguely conceived goal serves as an initial impetus to undertake a path of study. As he moves along that path the student's conception of the goal may change in accordance with what he has newly learned. The

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altered goal now reflects back and influences the further course of study. In most cases this is a subtle process of which the student may not be explicitly aware. We should be aware of it just as we should be aware of some reasons for learning that may not have occurred to a particular student.

But wait! Isn't this concern with motivation a tempest in a teapot? After all, for some time now everyone has known that mathematics is the language in which science describes and teaches us to manipulate the world. Computers are taking over more and more of the day to day tasks of the commercial world in which our students will earn a living. Most of these students are already convinced of the need to understand at least some mathematics.

Whenever we use one thing to motivate the study of a second we are also tacitly asserting that the first is more important than the second. This is implicit in the 'for-the-sake-of'. It would be foolish to act as though future prosperity is not or should not be important to students, but it is just as foolish to permit it to stand as the only or even the most important source of motivation. In the final analysis the conception of mathematics as a 'mere tool' is not only philosophically naive, it is debilitating to mathematics and alienating to students.

The alienation is rooted in a belief in a clearcut separation between the real world and the mathematical tools with which science describes and manipulates it. Students see that in order to function successfully in the world they must master those tools but they don't see an *essential* connection between the two. (This is very clear in the belief that reality would 'be the same' without mathematics.) Consequently they see the mathematics as a mere obstacle.

Some teachers accept this utilitarian view of mathematics and rely on it exclusively for motivation. (Sadly, this is often the case in the lower grades.) Others try to cultivate an appreciation for the beauty of mathematical results. These two views of mathematics (the utilitarian and the aesthetic) both tend to abet the belief in its isolation from the mainstream of day to day experience. I think this belief is false and unnecessary.

There is a large group of students to whom the utilitarian appeal seems crass and uninteresting but who are not so captivated by the beauty of mathematics as to want to learn it. These students can best be motivated by the teacher who has a rich understanding of the role mathematics plays in experience and is able to see its relationships to the most disparate human undertakings. It is part of the task of a philosophy of mathematics to generate such an understanding.

History. I am not concerned here with what we teach in courses on the history of mathematics nor with how or to what extent we incorporate historical remarks into our regular mathematics courses. These are issues which would require separate treatment even if, as I hope, some of the remarks which follow are relevant to them. My concern is to explicate the received view of the history of mathematics and to show its relevance.

Everyone knows that there is a regrettable ignorance of the history of mathematics. Thus it may seem misleading to speak of a 'received view'. But it is precisely the nature of the received view not to be explicit. Indeed, there has been much serious work done in the history of mathematics but it has barely influenced the mathematical community at large.

My synopsis of the 'received view': the history of mathematics is the story of what ideas emerged and what theorems were proved when and in what order. It may glance at relations to other subjects but its primary task is to show how mathematics has grown, how ideas have been synthesized and new ideas generated. On this view the reasons to study the history of mathematics are (1) to have a deeper understanding and appreciation of the results one has already learned, and (2) to realize that however beautiful and general present day mathematics is, it may develop further. A time will surely come when the theory of linear operators on Hilbert space will be routine, but that theory will retain its beauty. (At this point there should pass through the reader's mind the image of a faded photograph of a once beautiful woman; the image grows misty and evaporates.)

This history of mathematics is very different from most other histories. The differences are explicable in terms of the implicit conception of mathematics: a Euclidean system of eternal truths. The history of mathematics is the record of the successive unveilings of these truths to human intelligence. As such it is an unambiguous instance of *progress*. The most important difference between the history of mathematics and other histories is that it does not involve interpretation. The meaning of a mathematical result lies solely in its relations to the rest of mathematics. Thus, anachronism, a danger in most historical studies, is the procedure of choice in regard to mathematics. (Unless the timelessness of mathematical truths makes the use of "anachronism" improper.) The whole of mathematics, as revealed up to the present, is the exclusive standard by which the significance of a result can be legitimately assessed.

The severity of this conception no doubt accounts for the lack of widespread interest in or knowledge of the history of mathematics. However, it has been noticed in recent years that historical vignettes can add "human interest" to the study of mathematics. For example in [**33**], p. 336, we are informed that Newton was once sent to cut a hole in a barn door for cats to go in and out. Unfortunately such irrelevancies are probably counterproductive; they signal the student that the mathematics they accompany is indeed a bitter pill.

I hope that the reader objects to such an emasculated view of mathematics and its history. After all, we have provisionally agreed that an important function of mathematics is to generate methods for solving problems. This is a peculiarly human endeavor. While it is accurate to describe some animal behavior as problem solving, it appears that only man holds the problem before himself and reflects on how to solve it. Thus, to the extent that mathematics is the study or science of solving problems, it is as distinctively 'human' as literature or religion. (What this implies for the Euclidean conception remains to be unfolded.)

"But isn't it appropriate to separate mathematics proper from the essentially accidental human conditions of its discovery and use?" There is probably no point in trying to specify what this objection means by "mathematics proper". When it is pursued rigorously, the distinction which seems so clear on first glance melts into obscurity. What is more important is that if "mathematics proper" can carry a meaning in the objection then historical considerations are utterly irrelevant to understanding it.

In my view mathematics is like music in some respects; it is a creation of the human mind which generates its own standards of excellence or beauty rather than accepting them directly from other aspects of human experience. The quality of a particular solution, proof, or composition is primarily an internal affair. But a full understanding of mathematics (or music) can only come by seeing it in relation to the panorama of human possibilities. Thus the history of mathematics should not only record the development of mathematical ideas but it should also put in evidence the conceptions of mathematics which have prevailed at various times and exhibit how mathematics has participated in man's intellectual and practical life.

Now I can be more specific about the deficiencies of the received view of history.

1. It is blind to essential changes that have occurred in the role and conception of mathematics. The treatment of history in terms of evolution of important mathematical ideas (as for example in Eves [**6**] or Kramer [**23**]) needs to be supplemented by one based on a much broader view. For example, the conception of mathematics (and consequently its essential function) has changed very much from the classical Greek to the Christian and again from the Christian to the modern scientific period (since Galileo). An appreciation of these changes leads to a better understanding of the foundational crises which have dominated the landscape in philosophy of mathematics in this century. Notwithstanding certain superficial claims regarding the discovery of the incommensurability of the side and diagonal of a square, a 'crisis of the foundations' is not conceivable within the context of Greek mathematics.

When essential changes in the role of mathematics remain unnoticed it (mathematics) comes to be thought of as highly isolated from most human activity. This belief in its isolation is operative in the public at large and in our students. It is a major obstacle to understanding mathematics.

2. The received history of mathematics hardly recognizes the numerous false starts and incorrect proofs that have always been a part of mathematics. While there is good reason not to dwell on them, the cumulative effect of ignoring them is a picture of a science marching directly and inexorably to its goal. This helps perpetuate the mathematical mystique: that good people don't make mistakes. We know it is not so but the 'knowledge' is too abstract. We 'tell' our students that everyone makes mistakes; if we had a more accurate picture of the past our telling would carry more conviction.

3. Too seldom is explicit notice taken of the fact that what we 'cover' in one or two semesters of calculus is a distillation of over 150 years work of some very brilliant people. Of course, this reflects a pedagogical decision to provide the student with needed mathematical tools as quickly as possible. (A 'decision' which is often made on the basis of habit alone, without reflection, and is not always recognized as a decision.) But we owe it to our students to remind them that mathematics was not discovered inscribed in stone but is the creation of human beings like themselves struggling with problems.

4. The role of fashion and the influence of individuals is only rarely acknowledged. There are leaders, followers, and fads in mathematics just as there are in Shakespearian criticism, jazz, and clothing. This is a human phenomenon (or perhaps an all-too-human one). To recognize it not only helps to overcome the isolation of mathematics, it also makes it possible to look into the question: What qualities distinguish the mathematical leaders?

5. The history of mathematics is conceived for the most part as the record of the evolution of important mathematical ideas. This is a useful approach, especially for the mathematician. But again, because it remains wholly within the confines of mathematics itself, it lends credence to the myth of isolated mathematics. It is possible to look at the past in terms of what problems or types of problems were of compelling interest. This point of view affords more opportunity to examine the relationship between mathematics and other aspects of culture.

To take an example which may be farfetched, is it mere coincidence that the curbing of the unrestrained manipulation of infinite series coincided with a widespread rejection of Hegel's speculative philosophy? A less fantastic example is the importance for Plato of the problem of duplicating a cube. On this see van der Waerden [**36**], p. 138.

For an example from the more recent past, one must be struck by the fact that the logical foundations of mathematics seemed to crumble at nearly the same time as faith in cultural values and 'progress' was disintegrating.

6. The distinction between pure and applied mathematics and the idea of mathematics as a tool are read into the history of mathematics. This is closely related to the first point mentioned. It has the effect of rendering invisible the problematic character of the distinction (and of 'toolhood').

For example, we often read that mathematics began in ancient Egypt as applied mathematics. Indeed, this is shown by the very name: "geometry" means earth measurement. (Of course this name is Greek, not Egyptian, and reports the Greek understanding of what the Egyptians were doing.) To call an activity "applied mathematics" is to *appeal* to our distinction rather than to show that it is applicable. That distinction is based on the possibility of using mathematics as a tool with which to understand and manipulate nature. But this possibility exists only within the modern Western conception of nature, a conception which appears to be totally foreign to the ancient Egyptians, the ancient Greeks, and even the medieval Christians.

Philosophy. There are many who will claim that in passing from history to philosophy we pass from a light mist into a dense fog. Our historical remarks may have some relevance to the understanding and teaching of mathematics but what can we expect from philosophy? The chief interest in the philosophy of mathematics arose from the 'foundational crisis' provoked by the discovery of antinomies in set theory. This has been straightened out; indeed the effort has produced a robust mathematical logic. Mathematics has emerged from its crisis all the stronger and hardly seems in need of a 'philosophy'. There is something presumptuous about the very idea of a philosophy of mathematics. This prattle about Platonism, realism and the like—is there any point to it? Was Archimedes or Newton a philosopher? or Gauss? Philosophers can talk forever; mathematics *works.*

The differences between philosophy, history and mathematics seem to rest on the relative clarity of their problems. Philosophical problems are never clear; this is why positivists can do without philosophy. Historical problems run through a range from clear but trivial, e.g., who was the thirteenth president of the U.S.?, toward the philosophical, e.g., how has the abundance of natural resources in the U.S. influenced the evolution of our political system? In contrast to problems of this sort a mathematical problem is clear. It may be very difficult but we believe we will know when it's solved. There are proofs which are extremely complex (e.g., the classification of finite groups) but there seems to be a qualitative difference between complexity and the sort of obscurity which attends philosophical problems. Mathematical problems call for solutions; philosophical problems call for thought. An adequate philosophy of mathematics will show that this difference is not as sharp as I have portrayed it.

What is needed is a discussion that begins by examining what commonly passes as philosophy of mathematics. It should sift through the various problems and separate the trivial from the essential. Such a discussion must avoid the methodological naivety of thoughtlessly using the very ideas it examines. It should issue in a more profound understanding of mathematics, particularly of the role (for the most part invisible) that mathematics plays in our most basic conceptions of what is and what it means to question. In the present context this can be done only in the most sketchy form.

The issues of a philosophy of mathematics may be grouped under the following general headings: logical foundations, loss of certainty, the nature of proof, the relation of mathematical knowledge to the 'real world', and the 'ontological status' of mathematical objects such as numbers, sets, functions, etc. In particular the issue of logical foundations does not cover all of the philosophy of mathematics.

Preoccupation with foundational questions has forced the other issues out of view and has led to the trivialization of the philosophy of mathematics which is effective in the attitudes of most working mathematicians. Reuben Hersh (in [**18**]) describes the working mathematician as "a Platonist on weekdays and a formalist on Sundays." (By formalism he means "the philosophical position that much or all of pure mathematics is a meaningless game.") When mathematicians are unable to explain how their work has meaning they resort to calling it meaningless. But this is absurd; it's like suggesting that someone can not really recognize an old friend unless he can describe in a systematic way how he does it.

It seems obvious to me that there is no such thing as a meaningless game; it is equally obvious that pure mathematics is meaningful. As Hersh suggests, the choice between Platonism and formalism is too meager (it remains so when you include "intuitionism" and "logicism") and the mathematician should not be asked to make it. It is a philosophical problem to clarify the *phenomenon* of meaning in mathematics; an important one in my opinion.

Perhaps now that it is no longer *the* pressing issue to provide a logical foundation for mathematics we can ask about the meaning of this demand. What assumptions about the nature of mathematics and its relation to thought and experience in general are implicit in the demand for logical foundations? This is a question which can not be answered by producing a new definition of the word "set" or even of "foundation". It calls more for thought than for 'an answer'.

As one thinks about this problem there emerges a better understanding of problems in general; an understanding which can be useful to us in the classroom. At the same time that a problem asks for something (the 'unknown') it also specifies—sometimes explicitly but more often implicitly—a framework (which is not put into question) in terms of which or on the basis of which the solution is to be sought. Many problems in algebra and calculus are very simple in this respect. They are comparable to the problem of assembling a jigsaw puzzle when you know what the result should look like. The picture of the final result which appears on the box containing the puzzle is part of the framework within which one approaches the task. (The same is true of the assumption that the pieces do in fact fit together.)

If there is no picture on the box the problem is more difficult but is still possible. More of the framework is put into question if we are simply presented with a box of pieces and asked to assemble them so that they constitute a picture. Here the framework is reduced to the notions of what it means for pieces to fit together and what the word "picture" means.

The question about the foundations of mathematics appears to put into question something called the "nature of mathematics" on the basis of a framework of "thought and experience in general". The vagueness of the terms is an irremovable obstacle; it is an *essential* part of the problem. What is perhaps more important is that the more we think about it the clearer it becomes that we can not put into question the nature of mathematics without also disturbing the ideas of thought and experience in general.

This small insight does not solve the problem; it tends to shift it. It seems to me most fruitful to ask of the philosopher that he *make visible* the role that mathematics plays in our conception of thought and experience. This has a necessarily circular character which I think of as analogous to successive approximation. One begins with a crude idea of mathematics (the science of numbers and their relations for example) and a crude idea of experience (perception of material objects). But already here as soon as we have distinguishable objects a 'mathematical' idea, quantity, is implicit. Thinking about this leads to a little bit clearer idea of mathematics, etc.

The recognition that mathematics participates in experience to a much greater extent than we commonly think is perhaps the most important result of thinking about foundations. It has important ramifications for what I called above the relation of mathematical knowledge to the real world. (The 'real world' is that in which we eat and drink, cut grass, and hold most of our conversations. It is a world of material objects from which we look out to a world of ideas. It is a useful and fascinating metaphysical relic deserving of some veneration. It can also be a mischief maker.) Mathematics seems to grow from within itself without reference to the real world. And yet, again and again there arise applications for what was previously considered 'pure' mathematics. We believe that mathematics exists in two varieties, the pure and the applied. Davis and Hersh ([**4**], p. 86) quote Hardy's 'avowal of purity' and add "[it is a dominant ethos of twentieth-century mathematics] that the highest aspiration in mathematics is the aspiration to achieve a lasting work of art. If, on occasion, a beautiful piece of pure mathematics turns out to be useful, so much the better." Kline's book [**21**] on the other hand amounts to an impassioned argument that the proper roots and nourishment of mathematics lie in nature and applications.

We can arrive at some understanding of the issue of pure versus applied mathematics by first looking more carefully at what the distinction takes for granted. The quote from Davis and Hersh suggests a distinction between aesthetic-based and use-based evaluations. This is an extremely complicated situation from which I wish to draw out only one element: the aesthetic appreciation of an object (such as a proof of a theorem) is usually seen as an appreciation of the object itself; the object which is appreciated for its usefulness derives its value from that of the task for which it is useful. In the idea of applied mathematics there is a likeness to a tool.

This fact has its uses in the classroom. For many students it is very helpful to realize the similarity between how one approaches a mathematical problem and how one approaches a mechanical one. For example a comparison can be made between such methods of solving quadratic equations as trial and error or factoring and such methods of making holes as using a nail and hammer or using a push-drill. The quadratic formula is as useful in its way as an electric drill. Seeing the tool-aspect of mathematics in problem solving sometimes helps remove the aura of mystery which often accompanies the student's perception of mathematics.

But this conception of mathematics as a tool, which is fundamental to our idea of applied mathematics, still needs to be examined. Ordinarily we think of a tool as something inconspicuous which we use when the task demands it and then lay aside. For a time the role of language in articulating thought was likened to that of a tool but it was soon recognized that language plays a much more active role than the tool analogy suggests. In fact, I submit that the ordinary idea of a tool is one of those first-order approximations that function marvelously well in our day-to-day dealings but tend to collapse when asked to carry too much conceptual weight. Even on the level of purely mechanical tasks tools play an active role in the conception of problems. The mechanic's familiarity with what tools are available conditions his perception of a mechanical task just as the chess player's knowledge of how the pieces move conditions his perception of the chessboard.

The active role played by mathematical ideas in the 'perception' of applied problems is so great that I think the tool analogy is entirely misleading in this context. The distinction between pure and applied mathematics not only misses this fact; it covers it over. Certainly we can find 'pure' topics in mathematics and other topics which are used as passively as we ordinarily use a screwdriver. But we must not let that blind us to the very important role that mathematics plays in determining what counts for us as a problem and what would count as a solution. Once this role begins to be recognized the distinction between pure and applied mathematics begins to disintegrate along the common edge.

Wittgenstein puts this very neatly: "But what things are 'facts'? Do you believe that you can show what fact is meant by, e.g., pointing to it with your finger? Does that of itself clarify the part played by 'establishing' a fact?—Suppose it takes mathematics to define the *character* of what you are calling a 'fact'!... Why should not mathematics, instead of 'teaching us facts', create the forms of what we call facts?" (See [**38**], p. 381.)

Loss of certainty. The story of the loss of certainty is told by Morris Kline in his book *Mathematics: The Loss of Certainty*, [**21**]. He describes the emergence of the idea of God's mathematical design of nature during the 16th century as the result of tension between current religious belief and the reawakening interest in classical thought. Understanding of mathematical truth became tantamount to insight into God's design. This was a golden age for mathematics. Its results were as far beyond doubt as the possibility of a split between pure and applied aspects.

The remainder of the book describes the gradual loss of innocence. Mathematics reveals itself increasingly as a human creation and even as subject to human fallibility. It stood for a while as the last bastion of certainty, the final warrant for a belief in the possibility of knowledge, and then fell. From its remains there issued a squabble over foundations which has now subsided without genuine resolution. We mathematicians cannot hope to emerge from the shadow of Gödel's Theorems. "The efforts to eliminate possible contradictions and establish the consistency of the mathematical structures have thus far failed." ([**21**], p. 276).

Ostensibly this loss of certainty presents serious problems of a "philosophical" sort which are quite obviously related to mathematics. If indeed mathematics has served philosophers from Plato to Kant as a model for knowledge in general, then perhaps a change in the conception of mathematics should alter the understanding of that tradition. More relevant to us as mathematicians and teachers is the fact that our own understanding of what mathematics is has been conditioned by that same tradition (Plato to Kant). This results in a tension between our direct contact with mathematics and our understanding of it which is mediated by its recirculation through the tradition. This tension manifests itself in what Davis and Hersh call the Philosophical Plight of the Working Mathematician ([**4**], p. 321).

As far as the loss of certainty is concerned in itself (i.e., not as a historical-cultural phenomenon) it does not seem extraordinarily surprising or significant to me. I am far more puzzled by what 'absolute certainty' might mean than by the fact that mathematics doesn't offer it. It seems to me that there is a similarity between this historical event (of 450 years duration) and the debating tactic which builds and then destroys a straw man.

Mathematics is a human creation. That does not mean that it is arbitrary, but it does mean that it would be immodest to expect it to be 'certainly true' in the common sense of the phrase. It is incoherent to try and imagine mathematics as a source or body of absolutely certain knowledge. To say that a proof demonstrates the truth of a theorem is to say that it warrants the belief that the theorem will not at some future time be a party to a contradiction. The history of mathematics shows that standards of rigor in proof have varied and it affords examples of 'false proofs' which were widely accepted and 'correct proofs' which were widely rejected.

These examples indicate what we may call the "human element" in proof (in addition to the most obvious such, that proofs are made by humans). Proof is a form of mathematical discourse. It functions to unite mathematicians as practitioners of one mathematics. It is sometimes suggested that by formalizing proofs and building proof-checking machines we can reduce the influence of human fallibility on mathematical proofs. No doubt; but if this is seen as the first step of a surreptitious program to resurrect the 'absolute truth' of mathematics, it is doomed to failure. In fact, thinking about formal proofs and proofcheckers gives an interesting perspective on the human aspect.

First of all new, 'trivial', sources of error present themselves: errors in encoding the proof, errors from fatigue in dealing with the sheer complexity of some proofs, and even possible errors in the machine. Secondly and more importantly, a proof functions in mathematics only when it is *accepted* as a proof. This acceptance is a behavior of practicing mathematicians. Even if the verdict of a particular proofchecker comes to be accepted by the community of mathematicians they will reserve the right to reconsider and to admit nonformal proofs.

Fermat wrote that "the essence of proof is that it compels belief." To the extent that the compulsion operates via insight, (relatively) informal proofs will continue to play an important role in mathematics. Proofs that yield insight into the relevant concepts are more interesting and valuable to us as researchers and teachers than proofs that merely demonstrate the correctness of a result. We like a proof that brings out what seems to be essential. If the only available proof of a result is one that seems artificial or contrived it acts as an irritant. We keep looking and thinking. Instead of being able to move on, we are arrested. I mention these familiar facts only to emphasize that proof is not merely a system of links among various theorems, axioms, and definitions but also a system of discourse among people concerned with mathematics. As such it functions in a variety of ways.

Mathematical objects. Traditionally there has been much debate about the existence of mathematical objects such as numbers, categories, etc. This is closely related to the question of whether mathematical results are discovered or invented. It is extremely puzzling how these questions have managed to command as much attention as they have. In the first place it is hard to see what would stand or fall on the basis of a decision. Secondly it seems impossible to entertain the question seriously without making the tacit assumption that there is somehow a ground of 'absolute truth' from which to debate the issue. But there is no more reason to suppose such a ground than to imagine that we would recognize it if it came into view.

In **[9]** Gödel discussed the objects of set theory: "despite their remoteness from sense experience, we do have something like a perception also of the objects of set theory.... I don't see any reason why we should have less confidence in this kind of perception... than in sense perception, which induces us to build up physical theories and to expect that future sense perceptions will agree with them... ." He continues by pointing out that the objects of physical experience are no more immediately given than are the objects of mathematics. This passage asserts a 'Platonistic' position (especially a little beyond what I have quoted) but I like it for other reasons. First, because it makes quite clear that the objects of physical experience constitute the underlying model for our concept of existence. The more 'refined' models are produced by modification and analogy. Second, because it suggests that the object of physical experience, once we pass to the level of theoretical reflection, is just as problematic as the object of mathematics. We want to talk about the existence of mathematical objects because we want to make sense of the obvious fact that we are communicating when we discuss them. But why should their existence be something above and beyond this (possibility of) communication?

Empiricism. The last 'philosophical' issue I want to discuss is that of the difference between mathematics and the empirical sciences. Any reader who has come this far will know that I am not going to pretend to explain how things 'really stand'. One could do so only by begging the question. Rather I hope to draw into view some features of mathematics and empirical science that will render this distinction problematic.

The distinction appears to be founded on the obvious: empirical science has as its object the material reality which is common to and underlies all of human experience while the objects of mathematics are ideas. (The question of who makes the ideas may be left to the side.) In section two of **[25]** Lakatos describes this distinction as follows. In the traditional (Euclidean) model of mathematics, truth is injected at the top in the form of indubitable axioms and flows downward via deductive channels to particular theorems. On the other hand, empirical science does not begin with indubitable axioms; rather, what is beyond contest is at the bottom, the reality of hard facts. Falsehood flows upward to the theoretical assumptions. (If a theoretical prediction clashes with the facts the falsehood is retransmitted back to the theory which must then be modified.)

Lakatos maintains that Euclidean programs for mathematics have failed and that this failure has stimulated a tendency to see mathematics as more like the empirical sciences. More importantly a careful look at how mathematicians work and how they have been working for four hundred years makes clear that the Euclidean model is simply not descriptive of mathematics. (At most we can say that it is a model for the exposition of a mature part of mathematics; a model whose merits and liabilities are subject for legitimate debate.) In the last fifty years it has even ceased to be descriptive of how philosophers think about mathematics.

During the same time the notions of "hard facts" and "objective reality" have also become increasingly problematic. This has been provoked in part by reflections on developments in modern physics. But already the psychology of ordinary perception provides abundant evidence to undermine the idea of a reality which is immune to theoretical influence. Thus the conceptions of mathematics and empirical science have each been changing in the direction of the other.

The development of algebraic topology affords an example of a deep similarity between mathematics and empirical science. Eilenberg and Steenrod could only write their book, **[5]**, after a period in which ideas had been tried in a variety of combinations. Which combinations were best was decided by the leading problems but also by a developing internal criterion. The test of the axioms is always: do they capture the essentially relevant features of the guiding problems?

The situation seems to me entirely analogous to what has happened sometimes in the so-called human sciences. It often happens that a particular work captures, and thereby identifies, the essence of a certain chaotic development. (Very often this is a judgment which can be made only retrospectively.) Spengler's *Decline of the West* comes to mind; likewise Heidegger's *Being and*

Time. Such events are like a cold crystal in a rain cloud: they form a point of condensation. If the fog is thick enough the condensation spreads quickly and things soon look very different. The analogy goes further though. As time passes a shift takes place: instead of seeing *Being and Time* as the quintessential product of the intellectual climate of Germany in the twenties we begin to see Germany in the twenties as if through the lens of *Being and Time*.

Likewise, once the axioms proposed for a particular mathematical theory (a theory which has to have existed, perhaps in a disorganized way, *before* the axioms could have been formulated) are accepted (a complex phenomenon with a very definite sociological dimension), they begin to *define* that theory.

The development which I have just described within mathematics (and also, to some extent, within the 'human sciences') is very like what happens in empirical science. If we try to imagine a condition utterly without theory, then it seems as though 'the phenomena' simply present themselves. There is and can be no possible thought of illusion or deception. This condition I would refer to as the state of primordial experience. (I do not assert that such a condition has ever existed except perhaps as a theoretical idea.) Reality is a theoretical construct; it is the hypothetical *x of* which experience *is* experience. The empirical sciences depart from primordial experience; they aim to articulate it as a coherent, intelligible whole. (The word "whole" is important. Eventually, for example, the physical, biological, and ethical models of man must interlock perfectly.) Such ideas as object, material, force, organism, ecosystem, etc., are to the empirical sciences what the axioms are to a particular part of mathematics. They serve to organize experience and to make comparisons (and science) possible. As such they remain accountable to experience although this fact is obscured because they are also the basic elements in terms of which we elaborate that experience.

Puzzlement over the difference between mathematics and empirical science originates to a great extent in the fact that it is much more difficult for us to see the material object as a theoretical construct than, for example, the integer three. It is enhanced by the fact that we use sets of objects to found our formal definition of number and thereby give to objects a more 'fundamental' status. Finally, the fact that we write science in the language of mathematics influences us to think of mathematics as a free creation of the human spirit in contrast to science which is 'anchored to the facts'. I hope that my remarks have weakened this distinction.

Conclusion. In closing I must emphasize that I do not advocate teaching history or philosophy in mathematics courses. Rather, I maintain that the *teacher* needs to have a thorough acquaintance with them in order to teach more effectively. I will be criticized for not being more explicit about the mechanics of how understanding history and philosophy (in addition to mathematics) translates into more effective teaching. In the present context my response must be to appeal to an already familiar situation. When you teach a course which is not packaged in advance you need to know vastly more about the topic than you will ever discuss in detail. Otherwise there is no basis from which to choose which of many interesting beginnings will be most fruitful to pursue. Some have led to dead ends in the literature and some may lead up to the edge of current research. To be effective you need to have already explored these paths, not just the one that ends up being taken. Similarly, in our lower level courses we are teaching students how to solve certain types of problems. For the students the techniques at first seem remote from ordinary experience; for many of them mathematics is indeed a foreign language. We can not take the time to show them in detail the role mathematics has played in creating the world they take for granted and which seems to them so remote from the abstractions they are asked to master. But our awareness of that role can affect our approach at every turn; it manifests itself as an ease and breadth of understanding which is more convincing than speeches.

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